Control Law for an Intercept System

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A feedback control law is derived for a two-dimensional intercept system that combines in a step-by-step manner the closed-form solutions to a least-squares estimation of the target line speed and heading with a minimum control energy problem subject to a terminal intercept constraint. The major assumptions are that the pursuer possesses thrust modulation capabilities in addition to thrust vectoring, and that continuous measurements of the range coordinates are available for estimation purposes. Simulation results are included that indicate that good terminal accuracy is achieved when the estimation errors are small and illustrate the limitations on accuracy when fluctuations occur in the target speed and turn rate.

I. Introduction

AMONG the various alternatives that have been proposed for the proportional navigation guidance law in intercept problems, Deyst[1] applied optimal stochastic control theory, Ho[2] and Rajan[3] applied the theory of differential games, and Slater[4] and Nazaroff[5] took the perturbation point of view in applying linear optimal control theory. In this paper, a deterministic approach to a two-dimensional intercept problem is considered under the assumptions that the pursuer possesses thrust modulation capabilities in addition to the usual thrust vectoring, and that continuous measurements of the relative range coordinates are available. The thrust modulation capability facilitates the formulation of the control problem as a commutative bilinear system allowing for the application of some results recently obtained in Ref. 6. The philosophy for deriving the overall control law is basically the same as that used by the authors in Ref. 7 for the terminal control of a gliding parachute system in a nonuniform wind, viz. independent and easily computable solutions to the control and estimation problems, relative to a time interval $t_i \leq t \leq t_{i+1}$, are combined to define a step-by-step control-estimation sequence $i=0, 1, 2...$ which constitutes the closed-loop control law.

Following a statement of the problem in Sec. II, a least-squares estimation scheme is developed in Sec. III to estimate the target speed and relative heading. A closed-form solution to a minimum control energy problem with terminal constraint is obtained in Sec. IV, assuming the parameters for the target speed and heading are known. A modification is considered in Sec. V to include actuator dynamics for the thrust-vectoring engine. Simulation results are presented in Sec. VI that combine the estimation parameters from Sec. III with the controls obtained in Secs. IV and V for the closed-loop (feedback) control law.

II. Problem Statement

It is assumed for a high-speed pursuer and short initial range that the maneuvering of the vehicles can be restricted to a two-dimensional plane with the coordinates fixed in the pursuer. Denoting the angular rates of the pursuer and target with respect to a nonrotating reference frame by u_P and u_T, respectively, the kinematic equations of motions can be described by

$$\dot{x}_1 = -v_T \sin x_3 + x_2 u_P \ddagger$$

$$\dot{x}_2 = v_T \cos x_3 - x_1 u_P - v_P$$

$$\dot{x}_3 = u_T - u_P \tag{1}$$

where v_T and v_P are the line speeds of the target and pursuer relative to air, x_1 and x_2 are the horizontal and vertical distances of the target relative to the pursuer, and x_3 is the relative angle between the headings of the pursuer and the target.[5]

With a specified sequence of time instants t_i, $i=0, 1, 2...$, the estimation problem relative to the ith time interval, $t_i \leq t \leq t_{i+1}$, is to obtain estimates of the parameters $[v_T, x_3(t_i)]$ based on continuous measurements of the position coordinates $[x_1(t), x_2(t)]$ and an assumed knowledge of the quantities (v_P, u_P, u_T) on $[t_i, t_{i+1}]$.§ The control problem relative to the ith time instant t_i is to obtain a reasonable control strategy for $u_P(t)$ and $v_P(t)$, $t>t_i$, such that an intercept occurs at some future time $T>t_i$, i.e., $x_1(T)=x_2(T)=0$, assuming that the estimates of the parameters $[v_T, x_3(t_i)]$ are exact from a previous subinterval. This pair of estimation and control problems is solved anew on subsequent time intervals, $t_{i+1} \leq t \leq t_{i+2}$, etc., in defining the closed-loop control law. Precise statements and solutions to these problems are given in the following sections.

III. Least-Squares Estimation of (v_T, x_3)

Let $[t_0, t_1]$ denote a typical time interval over which continuous measurements of the range coordinates $[x_1(t), x_2(t)]$ are assumed given together with a knowledge of the quantities (v_P, u_P, u_T) on $[t_0, t_1]$. A deterministic least-squares estimate of the parameters $[v_T, x_3(t_0)]$ is considered here to obtain a simple closed-form solution to this aspect of the problem, thereby facilitating an easily implemented control law for the overall intercept problem as discussed earlier.

Denote the target speed v_T and initial heading $x_3(t_0)$ by the parameters v and a, i.e., $[v_T, x_3(t_0)] = (v, a)$. A least-squares

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‡Equation (1) follows from the well-known relation for the absolute velocity of a particle in a moving coordinate system, namely $\boldsymbol{v} = \dot{\boldsymbol{R}} + (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$, by defining a unit triad $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ at P so that $\boldsymbol{\rho} = x\boldsymbol{i} + y\boldsymbol{j}$, $\boldsymbol{R} = v_P \boldsymbol{j}$ and $\boldsymbol{\omega} = u_P \boldsymbol{k}$ at all times.

§Although pertinent to the problem at hand, the estimation of u_T will not be considered in this paper.

estimate of (a,v) results upon minimizing the functional

$$J(a,v) = \int_{t_0}^{t_1} \{\dot{x}_1(t) + v \sin[a+U(t)] - u_P x_2(t)\}^2 \mathrm{d}t + \int_{t_0}^{t_1} \{\dot{x}_2(t) - v\cos[a+U(t)] + v_P + u_P x_1(t)\}^2 \mathrm{d}t \quad (2)$$

where $U(t)$ is defined in terms of the target and pursuer turn rates, u_T and u_P, by

$$U(t) = \int_{t_0}^{t} [u_T - u_P(s)]\,\mathrm{d}s \quad (3)$$

A necessary condition for the minimization of J is that the partial derivatives of J with respect to a and v vanish:

$$\left.\frac{\partial J}{\partial a}\right|_{(a^*,v^*)} = 0 \qquad \left.\frac{\partial J}{\partial v}\right|_{(a^*,v^*)} = 0 \quad (4)$$

Observing that J is quadratic in v, the best estimate v^* can be uniquely determined in terms of a^*

$$v^* = -\frac{1}{t_1 - t_0}\{x_1(t_1)\sin[a^*+U(t_1)] - x_1(t_0)\sin a^* + x_2(t_0)\cos a^* - x_2(t_1)\cos[a^*+U(t_1)]\} - \int_{t_0}^{t_1}\left\{u_T\left[x_1(t)\cos[a^*+U(t)] + x_2(t)\sin[a^* + U(t)]\right] + v_P\cos[a^*+U(t)]\right\}\mathrm{d}t \quad (5)$$

Similarly,

$$0 = A\cos a^* + B\sin a^* \quad (6)$$

where

$$A = x_1(t_1)\cos U(t_1) + x_2(t_1)\sin U(t_1) - x_1(t_0) + \int_{t_0}^{t_1}\{[v_P + u_T x_1(t)]\sin U(t) - u_T x_2(t)\cos U(t)\}\mathrm{d}t \quad (7)$$

$$B = -x_1(t_1)\sin U(t_1) + x_2(t_1)\cos U(t_1) - x_2(t_0) + \int_{t_0}^{t_1}\{[v_P + u_T x_1(t)]\cos U(t) + u_T x_2(t)\sin U(t)\}\mathrm{d}t \quad (8)$$

From Eq. (6), we obtain

$$a^* = 2m\pi + \tan^{-1}\frac{A}{B} \qquad m = 0, \pm 1, \ldots \quad (9)$$

in which m is chosen so that $\partial^2 J/\partial a^2|_{(a^*,v^*)} > 0$. This is equivalent to

$$B\cos a^* > 0 \quad (10)$$

Following the solution for a^* from Eqs. (9) and (10), v^* is obtained from Eq. (5).

This solution for estimating $[v_T,\ x_3(t_0)]$ assumes a knowledge of u_T. Minimizing Eq. (2) over u_T will not lead to a simple closed-form solution and, therefore, another approach is necessary to estimate u_T. This is a separate issue that will be dealt with in a future paper using a new approach to parameter estimation problems recently developed in Ref. 8.

IV. Minimum Energy Control of the Intercept System

Let t_0 denote an arbitrary initial time with estimates of the parameters $[v_T,\ x_3(t_0)]$ assumed given from a previous time interval. Rather than consider $v_P(t)$ and $u_P(t)$ as independent control variables, a proportionality relation

$$v_P(t) = \gamma u_P(t) \quad (11)$$

is postulated with the proportionality parameter γ to be determined from the boundary conditions. The reason for this postulate is based on expediency in obtaining a closed-form solution to the following optimal control problem:

Minimize

$$J(u) = \int_{t_0}^{T} u^2(t)\,\mathrm{d}t\P \quad (12)$$

subject to the intercept condition

$$x_1(T) = x_2(T) = 0 \quad (13)$$

for some finite terminal time $T > t_0$ (a free time formulation), where the notation $u = u_P$ is used for simplicity.

The system (1) can be transformed into a commutative bilinear system by introducing the following auxiliary states:

$$x_4 = \sin x_3 \qquad x_5 = \cos x_3 \qquad x_6 = 1$$

That is, with these additional states, Eq. (1) can be written as

$$\dot{x} = Ax + Bxu \quad (14)$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (15a)$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (15b)$$

$$x(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ \sin x_3(t_0) \\ \cos x_3(t_0) \\ 1 \end{pmatrix} \quad (15c)$$

¶Although $J(u)$ is not the "physical" energy of the system, its minimization tends to reduce the control effort in satisfying the intercept condition (13).

It can be readily verified that $AB = BA$ so that Eq. (14) is a "commutative" bilinear system. The implication of this fact is that the optimal control solution to Eqs. (12) and (13), if it exists, is simply a constant function as has been shown in Theorem 3 of Ref. 6. The existence is contingent upon the terminal condition (13) being a reachable point for the given set of initial conditions. To examine this possibility, assume that v_T and u_T are constants so that Eq. (14) can be integrated explicitly as follows:

$$x_6(t) = 1 \quad x_5(t) = \cos x_3(t) \quad x_4(t) = \sin x_3(t) \quad x_3(t) = x_3(t_0) + u_T(t - t_0) - \int_{t_0}^{t} u(s)\,ds \tag{16}$$

and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t_0)\cos\int_{t_0}^{t} u(s)\,ds + x_2(t_0)\sin\int_{t_0}^{t} u(s)\,ds \\ -x_1(t_0)\sin\int_{t_0}^{t} u(s)\,ds + x_2(t_0)\cos\int_{t_0}^{t} u(s)\,ds \end{bmatrix} + v_T\int_{t_0}^{t} \begin{bmatrix} -\sin[x_3(t_0) - \int_{t_0}^{t} u\,ds + u_T(\xi - t_0)] \\ \cos[x_3(t_0) - \int_{t_0}^{t} u\,ds + u_T(\xi - t_0)] \end{bmatrix} d\xi$$

$$+\gamma \begin{bmatrix} \cos\int_{t_0}^{t} u\,ds - 1 \\ -\sin\int_{t_0}^{t} u\,ds \end{bmatrix} \tag{17}$$

We shall resolve the terminal constraint problem by considering the intercept angle as a parameter, then incorporate this solution with the minimum energy problem. Consideration should be given to two separate cases in which u_T is zero and nonzero, respectively.

Nonzero Angular Maneuver of the Target ($u_T = 0$)

The terminal constraint (13) on x_1 and x_2 requires (for some $T > t_0$):

$$0 = \begin{bmatrix} [x_1(t_0) + \gamma]\cos[u_T(T - t_0) + x_3(t_0) - \beta] + x_2(t_0)\sin[u_T(T - t_0) + x_3(t_0) - \beta] - \gamma \\ -[x_1(t_0) + \gamma]\sin[u_T(T - t_0) + x_3(t_0) - \beta] + x_2(t_0)\cos[u_T(T - t_0) + x_3(t_0) - \beta] \end{bmatrix} + \frac{v_T}{u_T}\begin{bmatrix} \cos\beta - \cos[u_T(T - t_0) - \beta] \\ \sin[u_T(T - t_0) - \beta] + \sin\beta \end{bmatrix} \tag{18}$$

where

$$\beta = x_3(t_0) - \int_{t_0}^{T} u(s)\,ds + u_T(T - t_0)$$

is defined as the intercept angle.

Therefore, the terminal constraint problem has been reduced to solving a pair of transcendental equations (18) for an appropriate set (γ, β, T). A solution often exists for this case in which the number of unknowns exceeds the number of equations. From Eq. (18) we obtain

$$\cos\beta = \cos x_3(t_0) - \frac{u_T}{v_T}x_1(t_0) + \frac{1}{\gamma}\left\{x_1(t_0)\cos x_3(t_0) + x_2(t_0)\sin x_3(t_0) - \frac{u_T}{2v_T}[x_1^2(t_0) + x_2^2(t_0)]\right\} \tag{19}$$

Given $\{x_1(t_0),\ x_2(t_0),\ x_3(t_0),\ u_T,\ v_T\}$, appropriate although nonunique values can easily be determined for γ and β such that Eq. (19) is satisfied. After γ and β are so determined, the intercept time T can be computed from Eq. (18) as follows

$$T = t_0 + \frac{1}{u_T}\left[2k\pi + \tan^{-1}\frac{F}{G}\right] \tag{20}$$

where

$$F = \gamma[\gamma + x_1(t_0)]\sin[\beta - x_3(t_0)] + \gamma x_2(t_0)\cos[\beta - x_3(t_0)] - \frac{v_T}{u_T}\{\gamma\sin\beta - [\gamma + x_1(t_0)]\sin x_3(t_0) + x_2(t_0)\cos x_3(t_0)\}$$

$$G = \gamma[\gamma + x_1(t_0)]\cos[\beta - x_3(t_0)] - \gamma x_2(t_0)\sin[\beta - x_3(t_0)] - \frac{v_T}{u_T}\left\{\gamma\cos\beta + [\gamma + x_1(t_0)]\cos x_3(t_0) + x_2(t_0) \times \sin x_3(t_0) - \frac{v_T}{u_T}\right\}$$

$k = 0, \pm 1, \ldots$ such that $T > t_0$

In summary, if a constant but nonzero angular maneuver of the target is assumed, then there exists a triple (γ, β, T) satisfying Eqs. (19) and (20) that solves the initial and terminal constraint problem (13) for every $[x_1(t_0),\ x_2(t_0),\ x_3(t_0)] \epsilon R^3$. The corresponding proper control action satisfies

$$\int_{t_0}^{T} u(s)\,ds = x_3(t_0) - \beta + u_T(T - t_0) \tag{21}$$

Zero Angular Maneuver of the Target ($u_T = 0$)

In a similar manner, the terminal constraint becomes

$$0 = \begin{bmatrix} [\gamma + x_1(t_0)]\cos[\beta - x_3(t_0)] - x_2(t_0)\sin[\beta - x_3(t_0)] - v_T(T-t_0)\sin\beta - \gamma \\ [\gamma + x_1(t_0)]\sin[\beta - x_3(t_0)] + x_2(t_0)\cos[\beta - x_3(t_0)] + v_T(T-t_0)\cos\beta \end{bmatrix}$$

where β is as defined in Eq. (18). This equation can be reduced to

$$\cos\beta = \cos x_3(t_0) + \frac{1}{\gamma}[x_1(t_0)\cos x_3(t_0) + x_2(t_0)\sin x_3(t_0)] \tag{22}$$

Finally, the intercept time T is determined by

$$T = t_0 + \frac{1}{v_T}\{[\gamma + x_1(t_0)]\sin x_3(t_0) - x_2(t_0)\cos x_3(t_0) - \gamma\sin\beta\} \tag{23}$$

A feasible T requires $T > t_0$, i.e., the term in the bracket must be positive. It can be shown (see Sec. 3.2 in Ref. 9) that this is true only for those initial conditions outside the region E defined by

$$E = \{(0,y,z)\epsilon R^3: \text{ either } y>0, z=(2k+1)\pi \text{ or } y<0,\ z=2k\pi;\ k=0,\pm 1,\dots\}$$

Therefore, if a zero angular maneuver of the target is assumed, then there exists a triple (γ, β, T) that solves the initial and terminal constraint problem (13) for every $(x_1(t_0), x_2(t_0), x_3(t_0))\ \epsilon R^3\backslash E$.**

It should be noticed that in the previous analyses the control function $u(t)$ has been eliminated for simplicity of computation. However, an admissible control that steers the pursuer to the target at T is associated with the triple (γ, β, T) through Eq. (21). Thus the set U_c of admissible controls is specified by

$$U_c = \left\{u\epsilon L^2([t_0,T],R): \int_{t_0}^{T} u(s)\,\mathrm{d}s = x_3(t_0) - \beta + u_T(T-t_0)\right\} \tag{24}$$

where β and T are determined by Eqs. (19) and (20) or (22) and (23).

With the set U_c of admissible controls furnished as in Eq. (24), we are ready to state the solution to the minimum energy problem (12). The following proposition is a direct consequence of Theorem 3 in Ref. 6.

Proposition 1

Given the system (14) and (15), these exists an optimal control $u^*\epsilon U_c$ which minimizes the cost (12) subject to the constraint (13) for each appropriate set of initial conditions $[x_1(t_0), x_2(t_0), x_3(t_0), u_T, v_T]$. This control is given by

$$u^*(t) = u_T + \frac{x_3(t_0) - \beta}{T - t_0} \tag{25}$$

where T and β are given in Eqs. (19) and (20) or (22) and (23).

V. Singularly Perturbed Problem

A more complex system model is considered in this section in which the pursuer turn rate is taken as the output of a first-order lag. This takes into account the practical situation in which the pursuer turn rate is furnished by a d.c. motor having first-order actuator dynamics. That is,

$$\epsilon\dot{u}(t) = -u(t) + u_0 \qquad \epsilon > 0 \tag{26}$$

where u is the real input. In the limiting case where ϵ approaches zero, this consideration is generally known as a singular perturbation problem.[10]

The cost functional in this case becomes

$$J(u_0) = \frac{1}{2}\int_{t_0}^{T} u_0^2(s)\,\mathrm{d}s \tag{27}$$

subject to the constraint

$$x_1(T) = x_2(T) = 0 \text{ for some } T > t_0 \tag{28}$$

By defining $z = u$, the system equations can be expressed by

$$\dot{x} = Ax + Bxz, \qquad \epsilon\dot{z} = -z + u_0 \qquad \epsilon > 0 \tag{29}$$

where A and B are the same as defined in Eq. (15).

Because it was shown in the last section that the terminal constraint problem has a solution when the control action satisfies Eq. (21), or in terms of the new state z

$$\int_{t_0}^{T} z(s)\,\mathrm{d}s = x_3(t_0) - \beta + u_T(T-t_0)$$

the terminal constraint problem (28) also has a solution provided

$$\frac{1}{\epsilon}\int_{t_0}^{T}\int_{t_0}^{t} u_0(s)e^{-(t-s)/\epsilon}\,\mathrm{d}s\,\mathrm{d}t = x_3(t_0) - \beta + u_T(T-t_0) + \epsilon z(t_0)[e^{-(T-t_0)/\epsilon} - 1] \tag{30}$$

Thus the set U_c of admissible controls for the problem (27) is the collection of inputs u_0 satisfying Eq. (30).

The existence of an optimal control $u_0^*\epsilon U_c$ which minimizes the cost (27) can be easily established, and by the Maximum Principle this optimal solution satisfies

$$u_0^*(t) = q(t) \tag{31}$$

and

$$\dot{p}(t) = -\partial H/\partial x = -(A' + B'z)p, \qquad \epsilon\dot{q}(t) = -\partial H/\partial z - x'B'p + q, \qquad \epsilon q(T) = 0 \tag{32}$$

In the latter expression (p,q) are costates corresponding to (x,z) and the prime denotes the matrix transpose operation. It can be easily checked that $\mathrm{d}/\mathrm{d}t(x'B'p) = 0$; hence $x'(t)B'p(t) = k$, a constant to be determined by the boundary conditions.

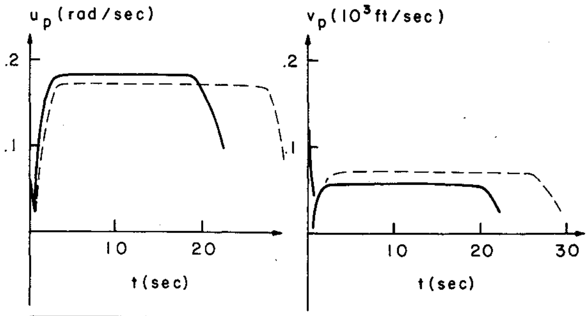
Substituting k into Eq. (32), q can be solved

$$q(t,\epsilon) = -k[e^{-(t-T)/\epsilon} - 1] = u_0^*(t) \tag{33}$$

Then from Eqs. (30) and (31), k is given by

$$k = \frac{1}{L}[x_3(t_0) - \beta + u_T(T-t_0) + \epsilon z(t_0)(e^{-(T-t_0)/\epsilon} - 1)] \tag{34}$$

**$A\backslash B$ denotes complement of B in A.



Trajs.	Initial Data		End Pt. Data	
	‖X(0)‖	β(0)	‖X(T)‖	β(T)
——	10000	57.29°	0	28.64°
----	8060	57.29°	0	28.64°

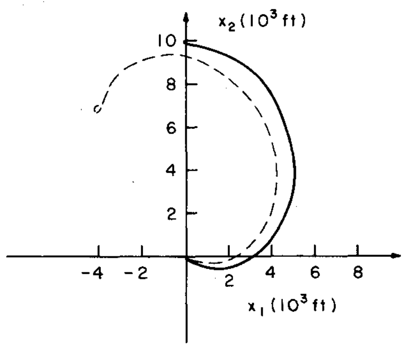
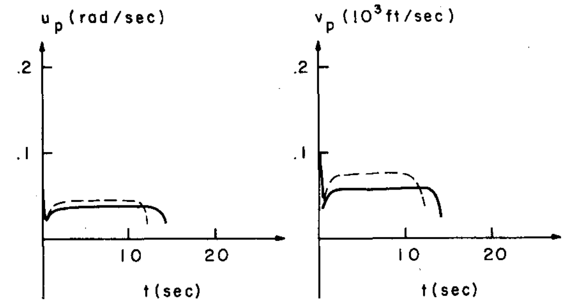


Fig. 1 Pursuer turn rate, speed, and relative trajectories.



Trajs.	Initial Data		End Pt. Data	
	‖X(0)‖	β(0)	‖X(T)‖	β(T)
——	10000	57.29°	0	28.64°
----	8250	57.29°	0	28.64°

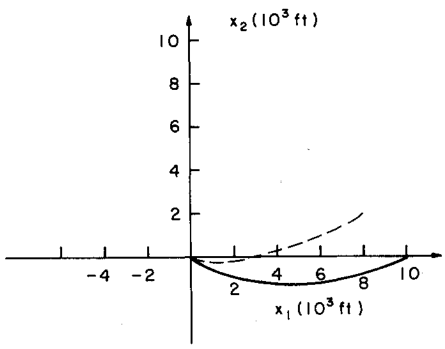


Fig. 2 Pursuer turn rate, speed, and relative trajectories.

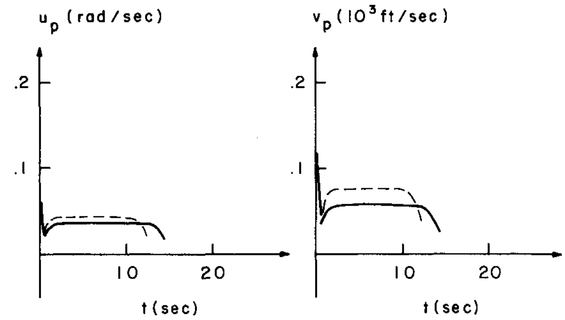
where

$$L=\epsilon[-1.5+\frac{1}{2}e^{(t_0-T)/\epsilon}+\frac{1}{2}e^{2(t_0-T)/\epsilon}]+T-t_0$$

These results are summarized in the next proposition.

Proposition 2

Given the system (31), there exists an optimal control $u_0^* \epsilon U_c$ that minimizes the cost (27) subject to the constraint (28) for



Trajs.	Initial Data		End Pt. Data	
	‖X(0)‖	β(0)	‖X(T)‖	β(T)
——	10000	57.29°	11.5	28.5°
----	8250	57.29°	14.5	28.6°

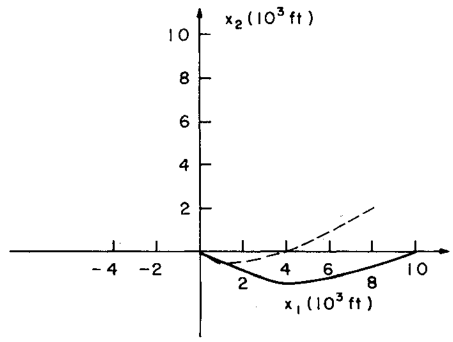


Fig. 3 Pursuer turn rate, speed, and relative trajectories.

each appropriate set of initial conditions. This control is given by Eqs. (33) and (34) where β and T are specified in Proposition 1.

It should be noticed that the singular perturbation comes into the problem (29) as ϵ approaches zero. This is clearly seen from the expressions of the optimal control (33) and (34), i.e.

$$\lim_{\epsilon\to 0} k(\epsilon)=\frac{1}{T-t_0}[x_3(t_0)-\beta+u_T(T-t_0)]$$

$$\lim_{\epsilon\to 0} u_0^*(t,\epsilon)=-\lim_{\epsilon\to 0} k(\epsilon)[e^{(t-T)/\epsilon}-1]$$

$$=u_T+\frac{x_3(t_0)-\beta}{T-t_0} \qquad t_0\le t\le T$$

which is exactly the same as that in Eq. (25). Therefore, for ϵ sufficiently small the solution to the optimal control problem (27) associated with a fourth-order system (29) can be approximated arbitrarily closely by the solution to the problem (12) associated with the third-order system (1).

In fact, not only the reduction of system order is shown here, but also an explicit solution to the optimization problem of the quadratic system†† is derived. This provides a great deal of potential to implement such a control law in practice because the first-order actuator dynamics have been included. Actually, this result can be generalized to include any higher order actuator dynamics as long as the constancy of the control area is sustained.

VI. Simulation Results

The least squares estimation scheme of Sec. III is combined with the optimal control law of Sec. IV and V to form a step-by-step feedback control of the intercept system for selected initial conditions and target turn rates. The initial target turn

††Note that Eq. (29) is no longer a bilinear system as defined in Sec. IV; instead, it is sometimes referred to as a quadratic system.

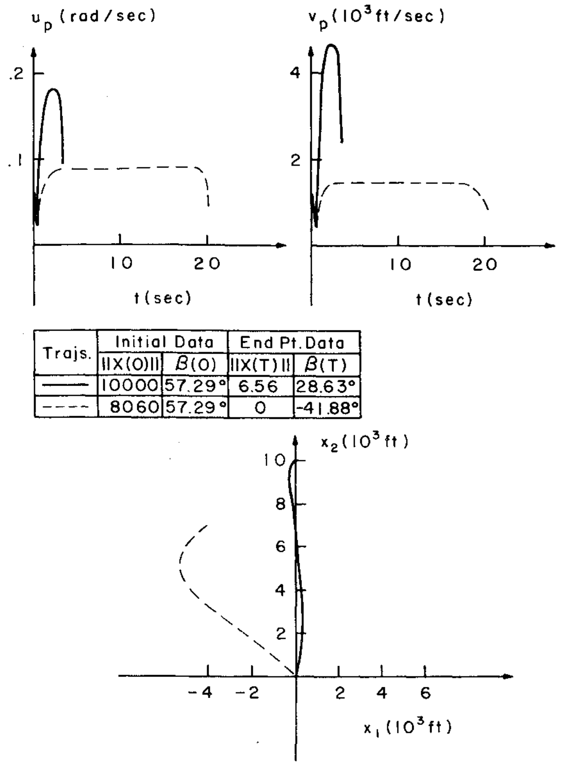


Fig. 4 Pursuer turn rate, speed, and relative trajectories.

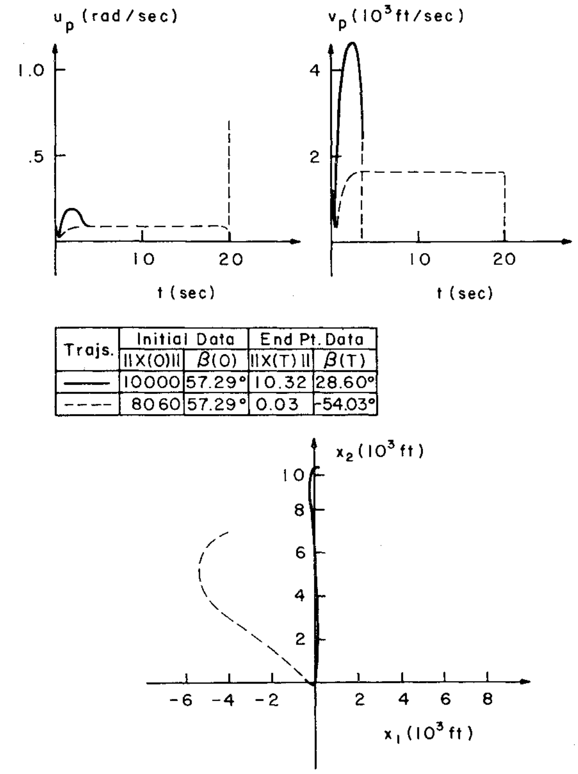


Fig. 5 Pursuer turn rate, speed, and relative trajectories.

rates are assumed to be 0.0 rad/s and 0.15 rad/s, while the target line speed is 1000 fps. In addition, two sets of sinusoidal fluctuations were imposed upon the target turn rate and line speed as listed in Table 1 to test the sensitivity of the system.

A time constant $\epsilon = 0.5$ s was assumed for the first-order actuator dynamics. During the first estimation interval of duration 0.5 s, the control effort was assumed to be $u(t) = 0.06e^{-2t}$ rad/s. Thereafter, the estimation intervals were determined by Eqs. (20) or (23) (depending on whether u_T is zero or nonzero) and consisted of a single interval when the estimation over $0 \leq t \leq 0.5$ was sufficiently accurate, as in Figs. 1, 2, and 4 where one interval of control led to an interception. Additional estimation and control intervals were needed for the perturbed target turn rates and line speed cases shown in Figs. 3 and 5. It is clear that in the case where the target turn rate and line speed are constant (Figs. 1, 2, and 4), the estimation of the target speed and the initial relative heading is exact and an ideal intercept is achieved in one interval as expected. For the case in which a fluctuation of either the target turn rate or its line speed is not known in advance (Figs. 3 and 5), the step-by-step control action is taken to offset the estimation error, which prolongs the expected intercept interval. Nevertheless, reasonable convergence occurs after a few loops of estimation in these particular cases.

Table 1 Actual target turn rates and line speeds

Figure no.	u_T (rad/s)	v_T (fps)
1	0.15	1000
2	0.00	1000
3	$-0.001 + 0.001(\cos 0.2t + \sin 0.2t)$	1000
4	0.00	1000
5	0.00	$980 + 20(\cos 0.2t + \sin 0.2t)$

Unknown fluctuations on the target turn rate of higher amplitude, e.g., $-0.005 + 0.005(\cos 0.2t + \sin 0.2t)$ were also considered, as well as a time-varying sawtooth target line speed, to test the effectiveness of the estimation scheme. Simulation results are relatively poor in these cases even though further estimation steps are called upon in trying to reduce the estimation error. This indicates the high sensitivity of the estimation scheme to deviations from the assumed constant values for the unknown parameters. It is suggested that a recursive estimator, which utilizes the first estimation data to initiate a secondary minimum variance estimation on either the target turn rate or line speed, may be worthy of consideration for future analyses.

VII. Concluding Remarks

The control law proposed here for an intercept problem is admittedly ad hoc. Separating the estimation and control problems, then combining their solutions in a step-by-step fashion, does not in any way constitute an optimal solution for the overall problem. Nevertheless, this approach does allow for a simple closed-form solution to the minimum control energy-terminal constraint problem and, it is believed, has enough elements of practicality to make it potentially attractive. The simulation results indicate good accuracy for an intercept when the estimation errors are small. At the same time, the simulations for various amplitudes of the fluctuations in the target speed and turn rate reflect the possible necessity for a higher order, or more sophisticated, estimation scheme to alleviate such fluctuation effects on terminal accuracy.

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